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The Two-Particle Hilbert Space:

Consider two particles 1 and 2 with corresponding Hilbert spaces V_1 and V_2. The two-particle system has a bigger Hilbert space $V_{1\otimes 2} = V_1 \otimes V_2$. It containts all possible states of the two-particle system. For each states of particle 1, particle 2 can assume any of its possible states, and vice versa. This should be reflected in the direct product of V_1 and V_2 that we have denoted as $V_1 \otimes V_2$.

Lets assume orthonormal bases $|n_1\rangle$ and $|n_2\rangle$ span V_1 and V_2 respectively. Then the direct product basis $|n_1\rangle \otimes |n_2\rangle$ (defined in $V_1 \otimes V_2$) spans the direct product Hilbert space.

The inner product of two direct product vectors is defined as following:

$$\langle x'_2| \otimes \langle x'_1| \; |x_1\rangle \otimes |x_2\rangle = \langle x'_1|x_1\rangle \langle x'_2|x_2\rangle =$$

$$\delta(x'_1 - x_1)\,\delta(x'_2 - x_2)$$

Depending on whether x_1 and x_2 are discrete or continuous labels, δ will be the Kronecker or Dirac delta function.

Note that the direct product of two vectors is very different from the inner or outer product. The inner product is defined for two vectors in the same Hilbert space and assigns a number to them. Also, the outer product is defined for two vectors belonging to the same Hilbert space and assigns a third vecter in the same Hilbert space to them.

However, direct product (also called tensor product) of two vectors is defined for vectors that belong to different Hilbert space and the result is a vector in a third (and bigger) Hilbert space.

Operators operation on vectors in $V_1 \otimes V_2$ can be constructed from those operating on V_1 and V_2 vectors. Lets consider two operators Ω_1^1 and Ω_2^2 operating on vectors belonging to V_1 and V_2 respectively. Here the subscript refers to the particle and the superscript to the Hilbert space. The direct product of Ω_1^1 and Ω_2^2 operates on vectors in $V_1 \otimes V_2$ according to:

$$\Omega_1^1 \otimes \Omega_2^2 \; |x_1\rangle \otimes |x_2\rangle = \Omega_1^1 |x_1\rangle \otimes \Omega_2^2 |x_2\rangle$$

Operation of Ω_1 on vectors in $V_1 \otimes V_2$ is:

$$\Omega_1^{1\otimes 2} \; |x_1\rangle \otimes |x_2\rangle = \Omega_1^1 \otimes \underbrace{I_2^2}_{\text{identity operator}} \; |x_1\rangle \otimes |x_2\rangle = \Omega_1^1 |x_1\rangle \otimes |x_2\rangle$$

This ensures that Ω_1 does not change the state of particle 2. Similarly:

$$\Omega_2^{1\otimes 2} |x_1\rangle \otimes |x_2\rangle = I_1^1 \otimes \Omega_2^2 |x_1\rangle \otimes |x_2\rangle = |x_1\rangle \otimes \Omega_2^2 |x_2\rangle$$

The matrix representation of $\Omega_1^1 \otimes \Omega_2^2$ is obtained from the matrix representations of Ω_1 and Ω_2:

$$\langle x_2'| \otimes \langle x_1'| \Omega_1^1 \otimes \Omega_2^2 |x_1\rangle \otimes |x_2\rangle = \langle x_1'|\Omega_1^1|x_1\rangle \langle x_2'|\Omega_2^2|x_2\rangle$$

It is the direct (or tensor) product of Ω_1 and Ω_2 matrix representations.

As an example, lets consider two particles with two possible states $|+\rangle$, $|-\rangle$ available to them. The two Hilbert spaces V_1 and V_2 are two dimensional each. There are two operators A_1^1 and B_2^2 (essentially 2x2 matrices) operation on vectors in V_1 and V_2 respectively. Defining:

$$|+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We have:

$$A_1^1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad a = \langle +| A_1^1 |+\rangle \quad b = \langle +| A_1^1 |-\rangle$$

$$c = \langle -| A_1^1 |+\rangle \quad d = \langle -| A_1^1 |-\rangle$$

$$B_2^2 = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \qquad e = \langle +| B_2^2 |+\rangle \quad f = \langle +| B_2^2 |-\rangle$$

$$g = \langle -| B_2^2 |+\rangle \quad h = \langle -| B_2^2 |-\rangle$$

We now want to find matrix representation of $A_1^{1\otimes 2}$. An orthonormal basis for $V_1 \otimes V_2$ is:

$$|+\rangle \otimes |+\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad |+\rangle \otimes |-\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad |-\rangle \otimes |+\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$|-\rangle \otimes |-\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The diagonal entries of $A_1^{1\otimes 2}$ are found from:

$$\langle +| \otimes \langle +| A_1^{1\otimes 2} |+\rangle \otimes |+\rangle = \langle +| A_1^1 |+\rangle \langle +| I_2^2 |+\rangle = a$$

$$\langle -| \otimes \langle +| A_1^{1\otimes 2} |+\rangle \otimes |-\rangle = \langle +| A_1^1 |+\rangle \langle -| I_2^2 |-\rangle = a$$

$$\langle + | \otimes \langle - | A_1^{1\otimes 2} | - \rangle \otimes | + \rangle = \langle - | A_1^1 | - \rangle \langle + | I_2^2 | + \rangle = d$$

$$\langle - | \otimes \langle - | A_1^{1\otimes 2} | - \rangle \otimes | - \rangle = \langle - | A_1^1 | - \rangle \langle - | I_2^2 | - \rangle = d$$

For the off-diagonal entries we have:

$$\langle - | \otimes \langle + | A_1^{1\otimes 2} | + \rangle \otimes | + \rangle = \langle + | A_1^1 | + \rangle \cancelto{0}{\langle - | I_2^2 | + \rangle} = 0$$

$$\langle + | \otimes \langle - | A_1^{1\otimes 2} | + \rangle \otimes | + \rangle = \langle - | A_1^1 | + \rangle \langle + | I_2^2 | + \rangle = C$$

$$\langle - | \otimes \langle - | A_1^{1\otimes 2} | + \rangle \otimes | + \rangle = \langle - | A_1^1 | + \rangle \cancelto{0}{\langle - | I_2^2 | + \rangle} = 0$$

$$\langle + | \otimes \langle + | A_1^{1\otimes 2} | + \rangle \otimes | - \rangle = \langle + | A_1^1 | + \rangle \cancelto{0}{\langle + | I_2 | - \rangle} = 0$$

$$\langle + | \otimes \langle - | A_1^{1\otimes 2} | + \rangle \otimes | - \rangle = \langle - | A_1^1 | + \rangle \cancelto{0}{\langle + | I_2^2 | - \rangle} = 0$$

$$\langle - | \otimes \langle - | A_1^{1\otimes 2} | + \rangle \otimes | - \rangle = \langle - | A_1^1 | + \rangle \langle - | I_2 | - \rangle = C$$

$$\langle + | \otimes \langle + | A_1^{1\otimes 2} | - \rangle \otimes | + \rangle = \langle + | A_1^1 | - \rangle \langle + | I_2 | + \rangle = b$$

$$\langle - | \otimes \langle + | A_1^{1\otimes 2} | - \rangle \otimes | + \rangle = \langle + | A_1^1 | - \rangle \cancelto{0}{\langle - | I_2 | + \rangle} = 0$$

$$\langle - | \otimes \langle - | A_1^{1\otimes 2} | - \rangle \otimes | + \rangle = \langle - | A_1^1 | - \rangle \cancelto{0}{\langle - | I_2 | + \rangle} = 0$$

$$\langle + | \otimes \langle + | A_1^{1\otimes 2} | - \rangle \otimes | - \rangle = \langle + | A_1^1 | - \rangle \cancelto{0}{\langle + | I_2^2 | - \rangle} = 0$$

$$\langle - | \otimes \langle + | A_1^{1\otimes 2} | - \rangle \otimes | - \rangle = \langle + | A_1^1 | - \rangle \langle - | I_2 | - \rangle = b$$

$$\langle + | \otimes \langle - | A_1^{1\otimes 2} | - \rangle \otimes | - \rangle = \langle - | A_1^1 | - \rangle \cancelto{0}{\langle + | I_2 | - \rangle} = 0$$

This results in:

$$A_1^{1\otimes 2} = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}$$

Similarly, it can be shown that:

$$B_2^{1\otimes 2} = \begin{bmatrix} e & f & 0 & 0 \\ g & h & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{bmatrix}$$

And:

$$A_1^1 \otimes B_2^2 = \begin{bmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{bmatrix}$$

Note that, as expected, dimensionality of $V_1 \otimes V_2$ comes from the product of dimensions of V_1 and V_2. This is different from direct sum of vector spaces where $\dim(V_1 \oplus V_2)$ is $\dim V_1 + \dim V_2$.